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LETTER TO THE EDITOR

ϵ expansion for directed animals

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Abstract. We introduce the problem of directed lattice animals and show that they are highly anisotropic with two different correlation lengths ξ_{\parallel} and ξ_{\perp} , parallel and perpendicular to the privileged direction. We introduce a field theory and calculate the critical exponents to first order in $\epsilon = d_c - d$ where the upper critical dimension, d_c , is 7.

The problem of directed percolation (Broadbent and Hammersley 1957) is a variant of isotropic bond percolation that has received considerable attention recently (Blease 1977, Obukhov 1980, Cardy and Sugar 1980, Dhar and Barma 1981, Kinzel and Yeomans 1981, Redner 1981). In directed percolation, bonds on a lattice are randomly occupied by diodes (rather than resistors) which always point along the positive direction of the lattice. Two sites are considered to be in the same cluster if there is a continuous path between them, or between a third site and each of them, along the positive sense of the diodes.

A problem related to isotropic percolation is the statistics of lattice animals (Domb 1976, Stauffer 1978a, b, Lubensky and Isaacson 1979). This is a purely geometrical problem, closely related to the configurational statistics of dilute branched polymers, in which one counts $A(n)$, the number of clusters per site on a lattice containing n bonds. For large n ,

$$A(n) \sim n^{-\theta} \lambda^n \quad (1)$$

where θ is a critical exponent and λ is a number related to the coordination number of the lattice. In this letter we shall consider the statistics of directed lattice animals (or branched polymers) which start from a single seed and in which all bonds point in the positive direction, as shown in figure 1.

As in the case of directed percolation (Dhar and Barma 1981, Domany and Kinzel 1981, Kinzel and Yeomans 1981, Klein and Kinzel 1981), we find that directed animals are highly anisotropic with correlation lengths ξ_{\parallel} and ξ_{\perp} , parallel and perpendicular to the privileged direction of diode orientation, satisfying

$$\xi_{\parallel} \sim n^{\eta} \quad (2a)$$

$$\xi_{\perp} \sim n^{\nu_{\perp}}. \quad (2b)$$

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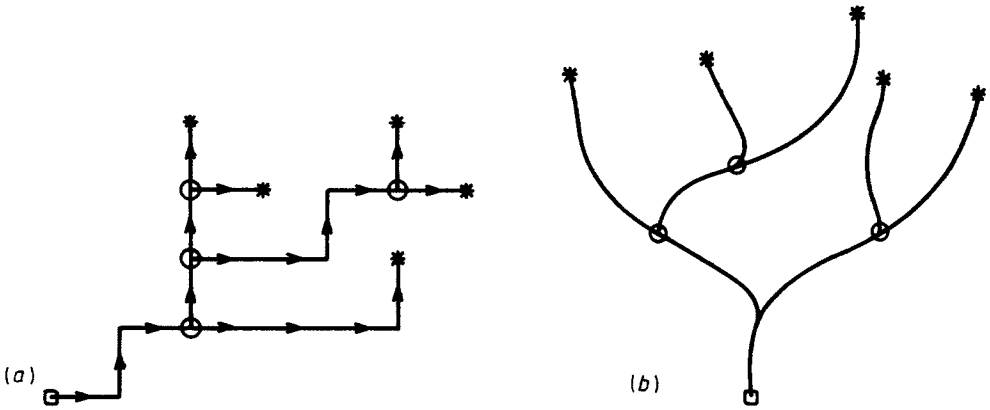


Figure 1. (a) A directed animal on a two-dimensional square lattice. The privileged direction is the (1, 1) direction. (b) A branched polymer which can only grow in the upward direction. Asterisks (*) mark the terminations, a square marks the root and circles mark the three point branches.

In mean field theory, $\nu_{\parallel} = \frac{1}{2}$ and $\nu_{\perp} = \frac{1}{4}$ and the upper critical dimension d_c is 7. Using a field theory similar to that developed for isotropic animals (Lubensky and Isaacson 1979, 1981a, b, Lubensky and McKane 1981, Parisi and Sourlas 1981) we calculate ν_{\parallel} , ν_{\perp} , θ and the critical point exponents η_{\parallel} and η_{\perp} to first order in $\epsilon = 7 - d$.

Let $\mathbf{X} = (\mathbf{X}_{\perp}, X_{\parallel})$ be a point in a d -dimensional space. X_{\parallel} is the component of \mathbf{X} along the privileged direction (the (1, 1, 1) direction for a $3d$ cubic lattice). We introduce two fields $\bar{\psi}(\mathbf{X})$ and $\psi(\mathbf{X})$ and choose the convention that propagation between $\bar{\psi}(\mathbf{X}_{\perp}, X_{\parallel})$ and $\psi(\mathbf{X}'_{\perp}, X'_{\parallel})$ can only occur if $X'_{\parallel} > X_{\parallel}$. We introduce the directed bond element

$$K(\mathbf{X}, \mathbf{X}') = \begin{cases} K & \text{if } X'_{\parallel} < X_{\parallel} \text{ and } \mathbf{X} \text{ and } \mathbf{X}' \text{ are nearest neighbours} \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

and the Hamiltonian

$$H = \int_{\mathbf{X}\mathbf{X}'} \bar{\psi}(\mathbf{X}) r(\mathbf{X}\mathbf{X}') \psi(\mathbf{X}') - \frac{1}{2} w \int_{\mathbf{X}} \psi(\mathbf{X}) \bar{\psi}^2(\mathbf{X}) - \int h \psi(\mathbf{X}) - \int \bar{h} \bar{\psi}(\mathbf{X}) + u \int (\psi(\mathbf{X}) \bar{\psi}(\mathbf{X}))^2 \quad (4)$$

where u represents the effects of repulsive interactions and

$$r(\mathbf{X}\mathbf{X}') = \delta_{\mathbf{X}\mathbf{X}'} - K(\mathbf{X}\mathbf{X}'). \quad (5)$$

The Fourier transform of r for small q is

$$r(\mathbf{q}) = r + q_{\perp}^2 + i q_{\parallel} \quad (6a)$$

$$r = 1 - zK \quad (6b)$$

where z is the number of forward nearest neighbours (equal to d for a cubic lattice in d dimensions).

The partition function associated with H is the generating function for lattice animals, or dilute branched polymers, with no loops

$$Z = \text{Tr } e^{-H}$$

$$= \sum_{\substack{N_p, N_e \\ N_2, N_3}} \bar{h}^{N_p} h^{N_e} w^{N_2} K^{N_3} C(N_p, N_e, N_2, N_3) \tag{7a}$$

$$= \sum (\bar{h}h)^{N_p} (wh)^{N_2} K^{N_3} C(N_p, N_2, N_3) \tag{7b}$$

where N_p , N_e , N_2 and N_3 are respectively the number of polymers, terminations, bonds and three point branches. $C(N_p, N_e, N_2, N_3)$ is the number of polymer configurations with N_p polymers, N_e terminations, N_2 bonds and N_3 three branches. To obtain (7b) from (7a) we used the fact that $N_e = N_p + N_3$ (see figure 1). In order to study the statistics of a single polymer (animal) we need to take the dilute limit of equation (7b)

$$\Phi = \lim_{\bar{h} \rightarrow 0} \frac{1}{\Omega h \bar{h}} \ln Z = \sum_{N_2, N_3} (wh)^{N_2} K^{N_3} C(1, N_2, N_3) \tag{8a}$$

$$= \sum_{N_2} K^{N_2} A(N_2) \sim (K - K_c)^{\theta-1}$$

where Ω is the volume of the system.

The analysis of the mean field theory for this problem proceeds almost exactly as for the ARB₂ polymers of Lubensky and Isaacson (1981b). The mean field equations of state are

$$r\bar{Q} - \frac{1}{2}w\bar{Q}^2 - h + 2u\bar{Q}^2Q = 0 \tag{9a}$$

$$rQ - w\bar{Q}Q - \bar{h} + 2u\bar{Q}Q^2 = 0. \tag{9b}$$

Note that $Q \rightarrow 0$ as $\bar{h} \rightarrow 0$ so the effects of repulsive interactions do not appear in mean field theory in this limit. Using equations (7) and (8) we find

$$\Phi = (wh)^{-1} [r - (r^2 - 2wh)^{1/2}] - 1 \tag{10}$$

so that $\Theta_{MF} = \frac{3}{2}$. The critical point is at $r_c = \sqrt{2wh}$.

To study spatial correlations and fluctuations we introduce the shifted order parameters $\phi = \psi - Q$ and $\bar{\phi} = \bar{\psi} - \bar{Q}$ to yield an effective Hamiltonian

$$H = \int_q \phi(q) t(q) \bar{\phi}(-q) + \frac{1}{2} T \int_q \phi(q) \phi(-q) - \frac{1}{2} w \int_{q_1, q_2, q_3} \phi(q_1) \bar{\phi}(q_2) \bar{\phi}(q_3) \delta^d(q_1 + q_2 + q_3) \tag{11}$$

in the dilute limit, where

$$t(q) = r - w\bar{Q} + q^2 + iq = t + q^2 + iq \tag{12a}$$

$$T = 2u\bar{Q}^2. \tag{12b}$$

Thus from equations (9) and (11) we find

$$G_{\bar{\psi}\psi}(q) = \langle \bar{\psi}(q) \psi(-q) \rangle = (\sqrt{r - 2wh} + q_{\perp}^2 + iq_{\parallel})^{-1} \tag{13a}$$

$$G_{\bar{\psi}\bar{\psi}}(q) = \langle \bar{\psi}(q) \bar{\psi}(-q) \rangle = -T |G_{\bar{\psi}\psi}(q)|^2. \tag{13b}$$

From this and the relation

$$G_{\bar{\psi}\bar{\psi}}(q=0) = \sum n^2 K^n A(n) \sim |K - K_c|^{-\gamma} \quad (14)$$

we conclude that

$$\gamma = \frac{1}{2} \quad \nu_{\perp} = \frac{1}{4} \quad \nu_{\parallel} = \frac{1}{2}. \quad (15)$$

The critical point exponents η_{\perp} and η_{\parallel} are defined via

$$G_{\bar{\psi}\bar{\psi}}(t=0, q_{\perp}, q_{\parallel}=0) = \frac{1}{q_{\perp}^{2-\eta_{\perp}}} \quad G_{\bar{\psi}\bar{\psi}}(t=0, q_{\perp}=0, q_{\parallel}) = \frac{1}{q_{\parallel}^{1-\eta_{\parallel}/2}} \quad (16)$$

so that $\eta_{\perp} = 0$ and $\eta_{\parallel} = 0$.

Note that we have defined γ in terms of $G_{\bar{\psi}\bar{\psi}}$. As with ARB₂ polymers, we could define a $\gamma_{\bar{\psi}\bar{\psi}}$ which would be $\frac{3}{2}$ from equations (9), (12) and (13). Note that the exponents in equation (15) result when h is held constant and t is varied. If $w\bar{Q}$ is fixed and t is varied, one obtains different exponents, as in the case of isotropic animals

$$\gamma_Q = 1 \quad \nu_{\perp Q} = \frac{1}{2} \quad \nu_{\parallel Q} = 1. \quad (17)$$

We now develop renormalisation group recursion relations in the usual way. The diagrams contributing to t , T and w are shown in figure 2. We obtain

$$dt/dl = (2 - \eta_{\perp})t + \frac{1}{4}w^2 TK_{d-1}(1+t)^{-2} \quad (18a)$$

$$dT/dl = (2 - \eta_{\perp})t - \frac{1}{8}w^2 T^2 K_{d-1}(1+t)^{-3} \quad (18b)$$

$$dw/dl = w[\frac{1}{2}(\varepsilon - 2 - \eta_{\parallel} - 2\eta_{\perp}) - \frac{1}{2}w^2 TK_{d-1}(1+t)^{-3}] \quad (18c)$$

$$\eta_{\perp} = -\frac{1}{12}w^2 TK_{d-1}(1+t)^{-3} \quad (18d)$$

$$\eta_{\parallel} = -\frac{1}{8}w^2 TK_{d-1}(1+t)^{-3}. \quad (18e)$$

From this we see that the upper critical dimension is 7 and the variable $g = \frac{1}{24}w^2 TK_{d-1}$, satisfying

$$dg/dl = g[\varepsilon - 18g(1+t)^{-3}], \quad (19)$$

controls the critical behaviour in $7 - \varepsilon$ dimensions.

From these equations we obtain

$$g^* = \frac{1}{18}\varepsilon \quad \eta_{\perp} = -\frac{1}{9}\varepsilon \quad \eta_{\parallel} = -\frac{1}{6}\varepsilon \\ \nu_{\parallel Q} = 1 + \frac{1}{4}\varepsilon \quad \nu_{\perp Q} = \frac{1}{2} + \frac{5}{36}\varepsilon \quad \gamma_Q = (2 - \eta)\nu_Q = 1 + \frac{1}{3}\varepsilon. \quad (20)$$

To obtain the exponents at constant wh , we have to solve the equation of state as for the case of isotropic animals. The analysis is the same as in Lubensky and Isaacson (1979, 1981b) and Lubensky and McKane (1981). We find that

$$\nu_{\perp} = \frac{\nu_Q}{1 + \gamma_Q} = \frac{1}{4} + \frac{\varepsilon}{36} \quad \nu_{\parallel} = \frac{\nu_Q}{1 + \gamma_Q} = \frac{1}{2} + \frac{\varepsilon}{24} \quad (21)$$

$$\gamma = \frac{\gamma_Q}{1 + \gamma_Q} = \frac{1}{2} + \frac{\varepsilon}{12} \quad \theta = 2 - \gamma = \frac{3}{2} - \frac{\varepsilon}{12}.$$

We should like to conclude with a few observations. First we note that the directed animals defined here have all bonds along the positive axes, no closed loops and only one root. One could imagine generalised directed polymers with many roots and

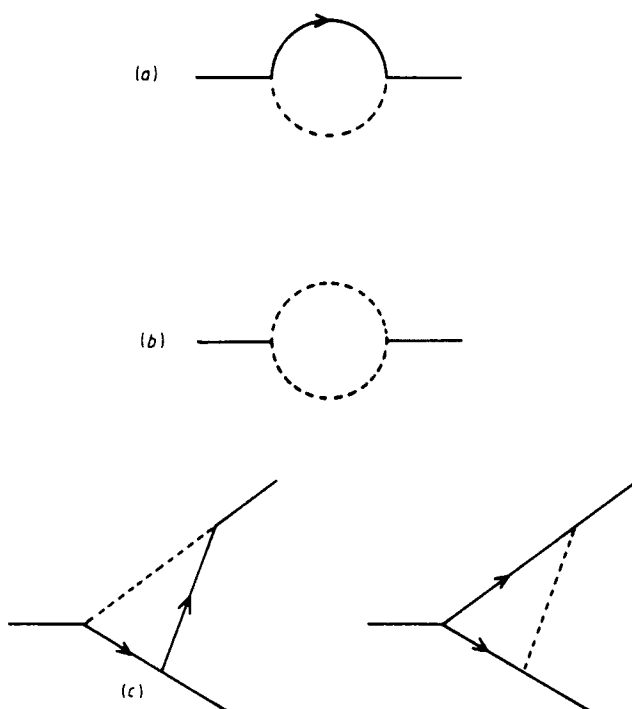


Figure 2. Diagrams contributing to $t(a)$, $T(b)$ and $w(c)$. The full line represents $G_{\vec{u}}$ and the broken line represents $G_{\vec{u},\vec{v}}$.

closed loops but still with all bonds in the forward direction. These are currently under study. Apparently the addition of loops does not change the universality class, at least in two dimensions (Nadal 1982). Secondly, one could imagine another class of animals that are more closely related to directed percolation. Here one first creates non-directed animals and then studies the statistics of directed animals in the interior of the non-directed animals. This problem appears to be considerably more complicated than the one stated here. Thirdly, Parisi and Sourlas (1981) have shown that the problem of isotropic animals is equivalent to that of the Yang-Lee edge singularity in two fewer dimensions. Because of the anisotropy we have not been able to find a similar relation for directed animals.

However, we note that the expressions (20) for ν_{\perp} and θ for directed animals to first order in $\varepsilon = 7 - d$ are the same as those for ν and θ for non-directed ARB₂ animals (Lubensky and Isaacson 1981b) to first order in $\varepsilon' = 8 - d$. Furthermore in two dimensions, ν_{\perp} is close to if not exactly $\frac{1}{2}$ (Nadal 1982) which is the value for ν in three dimensions (Parisi and Sourlas 1981). It is possible that ν_{\perp} and θ for directed animals in d dimensions are the same as ν and θ for non-directed ARB₂ animals in $d + 1$ dimensions for all $2 < d < 7$.

Finally, we comment on a possible amusing application of directed animals. It does not take much imagination to think of figure 1(b) as a green tree in a pasture. Thus one could imagine that directed animals might give an estimate of the height to width ratio of the tree as a function of its mass N

$$H/W \sim \xi_{\parallel}/\xi_{\perp} \sim N^{\eta-\nu_{\perp}}. \tag{22}$$

One could then imagine a forest of trees as a semi-dilute solution of directed animals and study H/W as a function of the number of trees per unit area σ . Arguments similar to those used to study polymer solutions (Daoud *et al* 1975) yield

$$H/W = N^{\nu_{\perp}} f(\sigma/\sigma^*) \quad (23)$$

where the crossover concentration σ^* satisfies

$$\xi^{d-1} \sigma^* \sim 1 \quad \text{or} \quad \sigma^* \sim N^{-(d-1)\nu_{\perp}}. \quad (24)$$

Note. While this manuscript was in preparation, we received a preprint from S Redner and Z R Yang in which they study the problem of directed animals. They point out that the mean field critical exponents ν_{\parallel} and ν_{\perp} are $\frac{1}{2}$ and $\frac{1}{4}$ and that the upper critical dimension is 7. They also evaluate $A(n)$ exactly in dimensions 2 to 8 for $n \leq 17$, $d = 2$ and $n \leq 6$ for $d = 8$ to obtain ν_{\parallel} , ν_{\perp} and θ . They find $\nu_{\perp} = \frac{1}{2}$ and $\theta = 0.53 \pm 0.01$ at $d = 2$ in agreement with the values for ν and θ for non-directed ARB₂ polymers in 3d.

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